51 Properties of Areas

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Nomenclature

A area units²
b base distance units
c distance to extreme fiber units
d separation distance units
h height distance units
i moment of inertia units²
J polar moment of inertia units⁴
L length units
P product of inertia units²
Q first moment of the area units²
r radius units
r radius of gyration units
S section modulus units²
V volume units
x distance in the x-direction units
y distance in the y-direction units

Symbols

θ angle degrees

Subscripts

c centroidal
o with respect to the origin

1. CENTROID OF AN AREA

The centroid of an area is analogous to the center of gravity of a homogeneous body. The centroid is often described as the point at which a thin homogeneous plate would balance. This definition, however, combines the definitions of centroid and center of gravity and implies that gravity is required to identify the centroid, which is not true.

The location of the centroid of an area bounded by the x- and y-axes and the mathematical function \( y = f(x) \) can be found by the integration method by using Eq. 51.1 through Eq. 51.4. The centroidal location depends only on the geometry of the area and is identified by the coordinates \( (x_c, y_c) \). Some references place a bar over the coordinates of the centroid to indicate an average point, such as \( \bar{x}, \bar{y} \).

\[
x_c = \frac{\int x \, dA}{A} \tag{51.1}
\]

\[
y_c = \frac{\int y \, dA}{A} \tag{51.2}
\]

\[
A = \int f(x) \, dx \tag{51.3}
\]

\[
dA = f(x) \, dx = g(y) \, dy \tag{51.4}
\]

The locations of the centroids of basic shapes, such as triangles and rectangles, are well known. The most common basic shapes have been included in App. 51.A. There should be no need to derive centroidal locations for these shapes by the integration method.

The centroid of a complex area can be found from Eq. 51.5 and Eq. 51.6 if the area can be divided into the basic shapes in App. 51.A. This process is simplified when all or most of the subareas adjoin the reference axis. Example 51.1 illustrates this method.

\[
x_c = \frac{\sum_i A_i x_{ci}}{\sum_i A_i} \tag{51.5}
\]

\[
y_c = \frac{\sum_i A_i y_{ci}}{\sum_i A_i} \tag{51.6}
\]

Example 51.1

An area is bounded by the x- and y-axes, the line \( x = 2 \), and the function \( y = e^x \). Find the x-component of the centroid.

\[x_c = \int x \, dA \]

\[y_c = \int y \, dA \]

\[A = \int f(x) \, dx \]

\[dA = f(x) \, dx = g(y) \, dy \]
Solution

First, use Eq. 51.3 to find the area.

\[ A = \int f(x) \, dx = \int_{x=0}^{x=2} e^{2x} \, dx \]
\[ = \frac{1}{2} e^{2x} \bigg|_0^2 = 27.3 - 0.5 = 26.8 \text{ units}^2 \]

Since \( y \) is a function of \( x \), \( dA \) must be expressed in terms of \( x \). From Eq. 51.4,

\[ dA = f(x) \, dx = e^{2x} \, dx \]

Finally, use Eq. 51.1 to find \( x_c \).

\[ x_c = \frac{\int x \, dA}{A} = \frac{1}{26.8} \int_{x=0}^{x=2} xe^{2x} \, dx \]
\[ = (\frac{1}{26.8}) \frac{1}{2} e^{2x} - \frac{1}{4} e^{2x} \bigg|_0^2 = 1.54 \text{ units} \]

Example 51.2

Find the \( y \)-coordinate of the centroid of the area shown.

Solution

The \( x \)-axis is the reference axis. The area is divided into basic shapes of a \( 1 \times 1 \) square, a \( 3 \times 8 \) rectangle, and a half-circle of radius 1. (The area could also be divided into \( 1 \times 4 \) and \( 3 \times 7 \) rectangles and the half-circle, but then the \( 3 \times 7 \) rectangle would not adjoin the \( x \)-axis.)

First, calculate the areas of the basic shapes. Notice that the half-circle area is negative since it represents a cut-out.

\[ A_1 = (1.0)(1.0) = 1.0 \text{ units}^2 \]
\[ A_2 = (3.0)(8.0) = 24.0 \text{ units}^2 \]
\[ A_3 = -\frac{1}{2} \pi r^2 = -\frac{1}{2} \pi (1.0)^2 = -1.57 \text{ units}^2 \]

Next, find the \( y \)-components of the centroids of the basic shapes. Most are found by inspection, but App. 51.A can be used for the half-circle. Notice that the centroidal location for the half-circle is positive.

\[ y_{c1} = 0.5 \text{ units} \]
\[ y_{c2} = 4.0 \text{ units} \]
\[ y_{c3} = 8.0 - 0.424 = 7.576 \text{ units} \]

Finally, use Eq. 51.6.

\[ y_c = \frac{\sum A_i y_{ci}}{\sum A_i} \]
\[ = \frac{(1.0)(0.5) + (24.0)(4.0) + (-1.57)(7.576)}{1.0 + 24.0 - 1.57} \]
\[ = 3.61 \text{ units} \]

2. FIRST MOMENT OF THE AREA

The quantity \( \int x \, dA \) is known as the first moment of the area or first area moment with respect to the \( y \)-axis. Similarly, \( \int y \, dA \) is known as the first moment of the area with respect to the \( x \)-axis. By rearranging Eq. 51.1 and Eq. 51.2, the first moment of the area can be calculated from the area and centroidal distance.

\[ Q_y = \int x \, dA = x_c A \quad 51.7 \]
\[ Q_x = \int y \, dA = y_c A \quad 51.8 \]

In basic engineering, the two primary applications of the first moment concept are to determine centroidal locations and shear stress distributions. In the latter application, the first moment of the area is known as the statical moment.
61 Alternating-Current Circuits

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Nomenclature

\(a\) inner radius \(m\)
\(a\) ratio of transformation –
\(b\) outer radius \(m\)
\(B\) magnetic flux density \(T\)
\(B\) susceptance \(S\)
\(BW\) bandwidth \(Hz\)
\(c\) speed of light \(m/s\)
\(C\) capacitance \(F\)
\(CF\) crest factor –

\(ERP\) effective radiated power \(W\)
\(f\) frequency \(Hz\)
\(FF\) form factor –
\(G\) conductance \(S\)
\(h\) hybrid parameter various
\(i, I\) current \(^1\) \(A\)
\(k\) coefficient of coupling –
\(k\) velocity factor –
\(L\) inductance \(H\)
\(n\) Steinmetz exponent –
\(N\) number of turns –
\(P\) power or power loss \(W\)
\(Q\) quality factor –
\(Q\) reactive power \(VAR\)
\(R\) resistance \(\Omega\)
\(S\) apparent power \(VA\)

\(SWR\) standing wave ratio –
\(t\) time \(s\)
\(T\) period \(s\)
\(U\) energy \(J\)
\(v\) velocity \(m/s\)
\(v, V\) voltage \(^2\) \(V\)
\(X\) reactance \(\Omega\)
\(y, Y\) admittance \(S\)
\(z, Z\) impedance \(\Omega\)

Symbols

\(\Gamma\) reflection coefficient –
\(\epsilon\) permittivity \(F/m\)
\(\eta\) efficiency –
\(\theta\) phase angle \(rad\)
\(\lambda\) wavelength \(m\)
\(\mu\) permeability \(H/m\)
\(\rho\) resistivity \(\Omega\cdot m\)
\(\sigma\) conductivity \(1/\Omega\cdot m\)
\(\tau\) time constant \(s\)
\(\phi\) impedance angle \(rad\)
\(\phi\) phase difference angle \(rad\)
\(\Phi\) magnetic flux \(Wb\)
\(\omega\) angular frequency \(rad/s\)

Subscripts

0 at resonance, characteristic
ave average
\(C\) capacitor
\(Cu\) copper
\(e\) eddy current or equivalent

\(^1\)The unsubscripted variables \(V\) and \(I\) in this chapter represent effective values.
\(^2\)See Ft. 1.
1. AC VOLTAGE

The term alternating waveform describes any symmetrical waveform, including square, sawtooth, triangular, and sinusoidal waves, whose polarity varies regularly with time. However, the term “AC” (i.e., alternating current) almost always means that the current is produced from the application of a sinusoidal voltage.\(^3\) Sinusoidal variables can be specified without loss of generality as either sines or cosines.\(^4\) If a sine waveform is used, Eq. 61.1 gives the instantaneous voltage as a function of time. \(V_m\) is the maximum value (also known as the amplitude) of the sinusoid. If \(V(t)\) is not zero at \(t = 0\), a phase angle, \(\theta\), must be used.\(^5\)

\[
v(t) = V_m \sin(\omega t + \theta) \tag{61.1}
\]

Figure 61.1 illustrates the period of the waveform, \(T\). (Since the horizontal axis corresponds to time and not distance, the waveform does not have a wavelength.)

![Figure 61.1 Sinusoidal Waveform with Phase Angle](image)

The frequency, \(f\), of the sinusoid is the reciprocal of the period in hertz (Hz). Angular frequency, \(\omega\), in rad/s can also be used. Table 61.1 describes standard electromagnetic frequency bands.

\[
f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{61.2}
\]

\[
\omega = 2\pi f = \frac{2\pi}{T} \tag{61.3}
\]

<table>
<thead>
<tr>
<th>Table 61.1 Standard Electromagnetic Frequency Band Designations</th>
<th>designation</th>
<th>meaning</th>
<th>frequency range</th>
</tr>
</thead>
<tbody>
<tr>
<td>ULF</td>
<td>ultra-low frequency</td>
<td>0.01 Hz to 10 Hz</td>
<td></td>
</tr>
<tr>
<td>ELF</td>
<td>extremely low frequency</td>
<td>10 Hz to 3 kHz</td>
<td></td>
</tr>
<tr>
<td>VLF</td>
<td>very low frequency</td>
<td>3 kHz to 30 kHz</td>
<td></td>
</tr>
<tr>
<td>LF</td>
<td>low frequency</td>
<td>30 kHz to 300 kHz</td>
<td></td>
</tr>
<tr>
<td>MF</td>
<td>medium frequency</td>
<td>300 kHz to 3 MHz</td>
<td></td>
</tr>
<tr>
<td>HF</td>
<td>high frequency</td>
<td>3 MHz to 30 MHz</td>
<td></td>
</tr>
<tr>
<td>VHF</td>
<td>very high frequency</td>
<td>30 MHz to 300 MHz</td>
<td></td>
</tr>
<tr>
<td>UHF</td>
<td>ultra-high frequency</td>
<td>300 MHz to 3 GHz</td>
<td></td>
</tr>
<tr>
<td>SHF</td>
<td>super high frequency</td>
<td>3 GHz to 30 GHz</td>
<td></td>
</tr>
<tr>
<td>EHF</td>
<td>extremely high frequency</td>
<td>30 GHz to 300 GHz</td>
<td></td>
</tr>
</tbody>
</table>

The imaginary exponentials \(e^{j\theta}\) and \(e^{-j\theta}\) can be combined to produce \(\sin \theta\) and \(\cos \theta\) terms. Therefore, it is not surprising that there are several equivalent methods of indicating a sinusoidal waveform in abbreviated form:

- trigonometric: \(V_m \sin(\omega t + \theta)\)
- exponential: \(V_m e^{j\theta}\)
- polar or phasor: \(V_m /j\)
- rectangular: \(V_r + jV_i\)

2. DEFECTS IN AC VOLTAGE

An AC voltage is rarely perfectly consistent in magnitude, phase, and frequency. The following are typical residential AC voltage defects.

- blackout: a complete failure that lasts more than one cycle
- brownout: a decrease in the steady-state voltage amplitude
- hunting: a frequency deviation effect resulting from irregular generation (i.e., uneven speed of the generator’s prime mover)
- line noise: interference that appears as small magnitude variations “riding” on the regular waveform caused by radio frequency and electromagnetic sources
- chronic overvoltage: an increase in the steady-state voltage amplitude that lasts for a long time
- sag (or dip): a disturbance (similar to a brownout but of shorter duration) that occurs when the line voltage drops below approximately 80% to 85% of its rated voltage by one or more cycles
1. ANALOG MODELS

In contrast to the usual case of using equations to predict the behavior of model physical systems, it is possible to build electrical circuits that simulate the solutions of equations, particularly ordinary linear differential equations with constant coefficients.1 These circuits are analog models. Voltages in such circuits correspond (are analogous) to unknown variables in the equations being investigated.

Rather than taking a breadboard, or experimental, approach in building such circuits, analog circuitry made for simulation has traditionally been implemented in an analog computer—a collection of components (op amps, potentiometers, oscilloscopes, etc.) in a common housing. The components are interconnected by programmable switch matrices. Although a few analog computers remain in use for teaching purposes in linear circuits, controls, and mechatronics labs, most modern simulation is performed digitally in Matlab, Simulink, and VisSim.

2. STRUCTURAL ELEMENTS

When voltage represents a variable, most of the standard mathematical operations (e.g., addition, multiplication, differentiation) with that variable can be performed electrically by passing the voltage through different components.2 For example, the change in voltage when a potentiometer is placed in the circuit is analogous to scalar multiplication. Table 71.1 lists the primary structural elements of an analog model and the operations they perform. High-gain direct current operational amplifiers are used to implement most of these elements.

The external input voltage to an analog circuit model corresponds to the forcing function of the nonhomogeneous differential equation being modeled. For example, a sinusoidal voltage will correspond to a sinusoidal forcing function, and so on. Such inputs are typically generated externally or as outputs from other analog model circuits.

Initial conditions are obtained by setting dedicated potentiometers on their corresponding integrators.

Example 71.1

An ideal operational amplifier is used as an inverting integrator. Its input represents a variable \( x(t) \) whose initial value is \( x(0) = 5 \). The variable is acted upon by a step input of height 3. (a) How would this be modeled? (b) What would be the output of the model at \( t = 4 \)?

Solution

(a) The symbolic representation of a model for this situation is

![Diagram](https://via.placeholder.com/150)

Table 71.1 Structural Elements of an Analog Model

<table>
<thead>
<tr>
<th>operation</th>
<th>symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar multiplication</td>
<td>( x \cdot K \cdot Kx )</td>
</tr>
<tr>
<td>negation</td>
<td>( x \rightarrow -x )</td>
</tr>
<tr>
<td>summation</td>
<td>( \frac{K_1 x_1 + K_2 x_2}{\Sigma} )</td>
</tr>
<tr>
<td>integration</td>
<td>( K(x + x(0)) )</td>
</tr>
<tr>
<td>input function</td>
<td>( f(t) )</td>
</tr>
</tbody>
</table>

---

1The mathematical equations, in turn, are generally used to model mechanical, fluid, thermal, and electrical systems.

2While possible, differentiation is seldom simulated due to the tendency differentiators have to develop electrical noise.
(b) Integrating a constant 3 results in $3t$. Adding in the initial value, the function is $x(t) = 3t + 5$. At $t = 4$, the function value is

$$x(4) = (3)(4) + 5 = 17$$

The inverted output is $-17$.

### 3. INDIRECT PROGRAMMING

The act of arranging the elements into a simulation model is known as indirect programming or mechanization. (The procedure given here assumes that there are no differentiating operational amplifiers in the mechanization of the differential equation.)

**step 1:** Normalize the differential equation so that the coefficient of the highest-order term is 1.0.

**step 2:** Solve for the negative of the highest-order term. (This accounts for the inversion performed by the final summing amplifier.)

**step 3:** Assume that the highest-order derivative exists. Route it through successive integrators to obtain lower-order raw derivatives.

**step 4:** Run raw derivatives through scaling potentiometers to obtain all required scaled derivatives.

**step 5:** Feed the forcing function and all scaled derivatives into the final summer.

**step 6:** Since the highest-order derivative is equal to the sum of all other terms (step 2), complete the circuit by connecting the output of the final summer to the point where the highest order derivative was assumed to exist in step 3.

**step 7:** Set the initial conditions on the potentiometers corresponding to the integrators used. Set the scaling potentiometers to the constant coefficients of their respective terms.

#### Example 71.2

Construct the analog model circuit that solves the differential equation.

$$2x'' + 0.32x' + 1.28x = 0.6$$

**Solution**

**step 1:** $x'' + 0.16x' + 0.64x = 0.3$

**step 2:** $-x'' = 0.16x' + 0.64x - 0.3$

**step 3:**

![Diagram](image)

#### Example 71.3

Construct the analog model circuit that simulates the following differential equation three times slower than its real time performance. Neglect initial conditions.

$$-x'' = 0.8x' + 0.3x - \sin(t)$$

**Solution**

The time-scaled differential equation is

$$-(3)^2 x'' = (3)(0.8)x' + 0.3x - \sin\left(\frac{T}{3}\right)$$