Mechanical Engineering Reference Manual
for the PE Exam

Twelfth Edition

Michael R. Lindeburg, PE

Professional Publications, Inc. • Belmont, CA

Topic X: Dynamics and Vibrations
Machine Design Depth

Chapter

54. Properties of Solid Bodies
55. Kinematics
56. Kinetics
57. Mechanisms and Power Transmission Systems
58. Vibrating Systems

For the most current information about the exam, visit **www.ppi2pass.com** regularly.

55 Kinematics

1. Introduction to Kinematics 55-1
2. Particles and Rigid Bodies 55-1
3. Coordinate Systems 55-1
4. Conventions of Representation 55-2
5. Linear Particle Motion 55-2
6. Distance and Speed 55-2
7. Uniform Motion 55-3
8. Uniform Acceleration 55-3
9. Linear Acceleration 55-4
10. Projectile Motion 55-4
11. Rotational Particle Motion 55-6
12. Relationship Between Linear and Rotational Variables 55-7
13. Normal Acceleration 55-7
14. Coriolis Acceleration 55-8
15. Particle Motion in Polar Coordinates 55-9
16. Relative Motion 55-9
17. Dependent Motion 55-11
18. General Plane Motion 55-11
19. Rotation About a Fixed Axis 55-12
20. Instantaneous Center of Acceleration . . . 55-13
21. Slider Rods 55-13
22. Slider-Crank Assemblies 55-13

Nomenclature

a	acceleration	ft/sec^2	m/s^2
d	distance	ft	m
g	gravitational acceleration	ft/sec^2	m/s^2
H	height	ft	m
l	length	ft	m
n	rotating speed	rpm	rpm
r	radius	ft	m
R	range	ft	m
R	earth's radius	ft	m
s	distance	ft	m
t	time	sec	s
T	flight time	sec	s
v	velocity	ft/sec	m/s
z	elevation	ft	m

Symbols

α	angular acceleration	rad/sec^2	rad/s^2
β	angle	deg	deg
γ	angle	deg	deg
θ	angular position	rad	rad
ϕ	angle or latitude	deg	deg
ω	angular velocity	rad/sec	rad/s

Subscripts

0	initial
ϕ	transverse
a	acceleration
c	Coriolis
H	to maximum altitude
n	normal
O	center
r	radial
t	tangential

1. INTRODUCTION TO KINEMATICS

Dynamics is the study of moving objects. The subject is divided into kinematics and kinetics. *Kinematics* is the study of a body's motion independent of the forces on the body. It is a study of the geometry of motion without consideration of the causes of motion. Kinematics deals only with relationships among position, velocity, acceleration, and time. (Kinetics is covered in Ch. 56.)

2. PARTICLES AND RIGID BODIES

Bodies in motion can be considered *particles* if rotation is absent or insignificant. Particles do not possess rotational kinetic energy. All parts of a particle have the same instantaneous displacement, velocity, and acceleration.

A *rigid body* does not deform when loaded and can be considered a combination of two or more particles that remain at a fixed, finite distance from each other. At any given instant, the parts (particles) of a rigid body can have different displacements, velocities, and accelerations.

3. COORDINATE SYSTEMS

The position of a particle is specified with reference to a *coordinate system*. The description takes the form of an ordered sequence $(q_1, q_2, q_3, \ldots)$ of numbers called *coordinates*. A coordinate can represent a position along an axis, as in the rectangular coordinate system, or it can represent an angle, as in the polar, cylindrical, and spherical coordinate systems.

In general, the number of *degrees of freedom* is equal to the number of coordinates required to completely specify the state of an object. If each of the coordinates is independent of the others, the coordinates are known as *holonomic coordinates*.

The state of a particle is completely determined by the particle's location. In three-dimensional space, the locations of particles in a system of m particles must be specified by $3m$ coordinates. However, the number of required coordinates can be reduced in certain cases. The position of each particle constrained to motion on a surface (i.e., on a two-dimensional system) can be specified by only two coordinates. A particle constrained to moving on a curved path requires only one coordinate.[1]

The state of a rigid body is a function of orientation as well as position. Six coordinates are required to specify the state: three for orientation and three for location.

4. CONVENTIONS OF REPRESENTATION

Consider the particle shown in Fig. 55.1. Its position (as well as its velocity and acceleration) can be specified in three primary forms: vector form, rectangular coordinate form, and unit vector form.

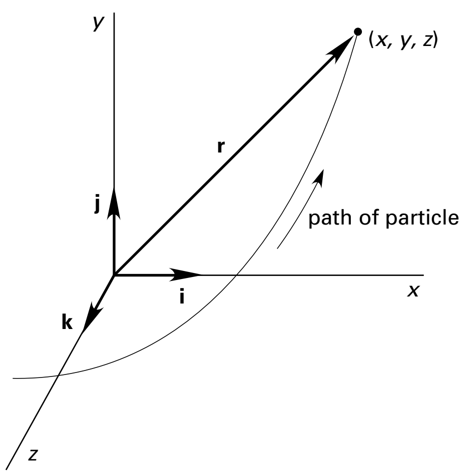

Figure 55.1 *Position of a Particle*

The vector form of the particle's position is $\mathbf{r}$, where the vector $\mathbf{r}$ has both magnitude and direction. The rectangular coordinate form is (x, y, z). The unit vector form is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad 55.1$$

5. LINEAR PARTICLE MOTION

A *linear system* is one in which particles move only in straight lines. (Another name is *rectilinear system*.) The relationships among position, velocity, and acceleration for a linear system are given by Eqs. 55.2 through 55.4. When values of t are substituted into these equations, the position, velocity, and acceleration are known as *instantaneous values*.

$$s(t) = \int \mathrm{v}(t)\,dt = \int \left(\int a(t)\,dt \right) dt \quad 55.2$$

$$\mathrm{v}(t) = \frac{ds(t)}{dt} = \int a(t)\,dt \quad 55.3$$

$$a(t) = \frac{d\mathrm{v}(t)}{dt} = \frac{d^2 s(t)}{dt^2} \quad 55.4$$

The average velocity and acceleration over a period from t_1 to t_2 are

$$\mathrm{v}_{\text{ave}} = \frac{\int_1^2 \mathrm{v}(t)\,dt}{t_2 - t_1} = \frac{s_2 - s_1}{t_2 - t_1} \quad 55.5$$

$$a_{\text{ave}} = \frac{\int_1^2 a(t)\,dt}{t_2 - t_1} = \frac{\mathrm{v}_2 - \mathrm{v}_1}{t_2 - t_1} \quad 55.6$$

Example 55.1

A particle is constrained to move along a straight line. The velocity and location are both zero at $t = 0$. The particle's velocity as a function of time is

$$\mathrm{v}(t) = 8t - 6t^2$$

(a) What are the acceleration and position functions?
(b) What is the instantaneous velocity at $t = 5$?

Solution

(a)

$$a(t) = \frac{d\,\mathrm{v}(t)}{dt} = \frac{d(8t - 6t^2)}{dt} = 8 - 12t$$

$$s(t) = \int \mathrm{v}(t)\,dt = \int (8t - 6t^2)\,dt = 4t^2 - 2t^3$$

(b) Substituting $t = 5$ into the $\mathrm{v}(t)$ function,

$$\mathrm{v}(5) = (8)(5) - (6)(5)^2 = -110 \quad \text{[backward]}$$

6. DISTANCE AND SPEED

The terms "displacement" and "distance" have different meanings in kinematics. *Displacement* (or *linear displacement*) is the net change in a particle's position as determined from the position function, $s(t)$. *Distance traveled* is the accumulated length of the path traveled during all direction reversals, and can be found by adding the path lengths covered during periods in which the velocity sign does not change. Thus, distance is always greater than or equal to displacement.

$$\text{displacement} = s(t_2) - s(t_1) \quad 55.7$$

[1]The curve can be a straight line, as in the case of a mass hanging on a spring and oscillating up and down. In this case, the coordinate will be a linear coordinate.

Similarly, "velocity" and "speed" have different meanings: *velocity* is a vector, having both magnitude and direction; *speed* is a scalar quantity, equal to the magnitude of velocity. When specifying speed, direction is not considered.

Example 55.2

What distance is traveled during the period $t = 0$ to $t = 6$ by the particle described in Ex. 55.1?

Solution

Start by determining when, if ever, the velocity becomes negative. (This can be done by inspection, graphically, or algebraically.) Solving for the roots of the velocity equation, the velocity changes from positive to negative at

$$t = \frac{4}{3}$$

The initial displacement is zero. From the position function, the position at $t = 4/3$ is

$$s\left(\frac{4}{3}\right) = (4)\left(\frac{4}{3}\right)^2 - (2)\left(\frac{4}{3}\right)^3 = 2.37$$

The displacement while the velocity is positive is

$$\begin{aligned}\Delta s &= s\left(\frac{4}{3}\right) - s(0) = 2.37 - 0\\ &= 2.37\end{aligned}$$

The position at $t = 6$ is

$$s(6) = (4)(6)^2 - (2)(6)^3 = -288$$

The displacement while the velocity is negative is

$$\begin{aligned}\Delta s &= s(6) - s\left(\frac{4}{3}\right) = -288 - 2.37\\ &= -290.37\end{aligned}$$

The total distance traveled is

$$2.37 + 290.37 = 292.74$$

7. UNIFORM MOTION

The term *uniform motion* means uniform velocity. The velocity is constant and the acceleration is zero. For a constant velocity system, the position function varies linearly with time.

$$s(t) = s_0 + \mathrm{v}t \qquad 55.8$$

$$\mathrm{v}(t) = \mathrm{v} \qquad 55.9$$

$$a(t) = 0 \qquad 55.10$$

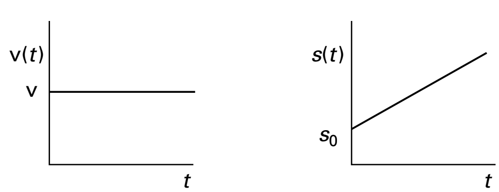

Figure 55.2 *Constant Velocity System*

8. UNIFORM ACCELERATION

The acceleration is constant in many cases. (Gravitational acceleration, where $a = g$, is a notable example.) If the acceleration is constant, the a term can be taken out of the integrals in Eqs. 55.2 and 55.3.

$$a(t) = a \qquad 55.11$$

$$\mathrm{v}(t) = a\int dt = \mathrm{v}_0 + at \qquad 55.12$$

$$s(t) = a\iint dt^2 = s_0 + \mathrm{v}_0 t + \tfrac{1}{2}at^2 \qquad 55.13$$

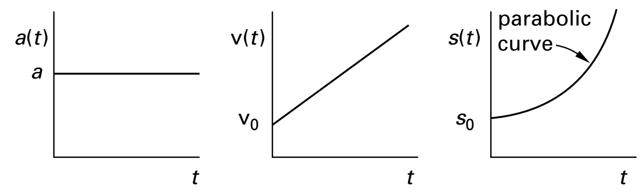

Figure 55.3 *Uniform Acceleration*

Table 55.1 summarizes the equations required to solve most uniform acceleration problems.

Table 55.1 *Uniform Acceleration Formulas*[a]

to find	given these	use this equation
a	$t, \mathrm{v}_0, \mathrm{v}$	$a = \dfrac{\mathrm{v} - \mathrm{v}_0}{t}$
a	t, v_0, s	$a = \dfrac{2s - 2\mathrm{v}_0 t}{t^2}$
a	$\mathrm{v}_0, \mathrm{v}, s$	$a = \dfrac{\mathrm{v}^2 - \mathrm{v}_0^2}{2s}$
s	t, a, v_0	$s = \mathrm{v}_0 t + \tfrac{1}{2}at^2$
s	$a, \mathrm{v}_0, \mathrm{v}$	$s = \dfrac{\mathrm{v}^2 - \mathrm{v}_0^2}{2a}$
s	$t, \mathrm{v}_0, \mathrm{v}$	$s = \tfrac{1}{2}t(\mathrm{v}_0 + \mathrm{v})$
t	$a, \mathrm{v}_0, \mathrm{v}$	$t = \dfrac{\mathrm{v} - \mathrm{v}_0}{a}$
t	a, v_0, s	$t = \dfrac{\sqrt{\mathrm{v}_0^2 + 2as} - \mathrm{v}_0}{a}$
t	$\mathrm{v}_0, \mathrm{v}, s$	$t = \dfrac{2s}{\mathrm{v}_0 + \mathrm{v}}$
v_0	t, a, v	$\mathrm{v}_0 = \mathrm{v} - at$
v_0	t, a, s	$\mathrm{v}_0 = \dfrac{s}{t} - \tfrac{1}{2}at$
v_0	a, v, s	$\mathrm{v}_0 = \sqrt{\mathrm{v}^2 - 2as}$
v	t, a, v_0	$\mathrm{v} = \mathrm{v}_0 + at$
v	a, v_0, s	$\mathrm{v} = \sqrt{\mathrm{v}_0^2 + 2as}$

[a]The table can be used for rotational problems by substituting α, ω, and θ for a, v, and s, respectively.

Example 55.3

A locomotive traveling at 80 kph locks its wheels and skids 95 m before coming to a complete stop. If the deceleration is constant, how many seconds will it take for the locomotive to come to a standstill?

Solution

First, convert the 80 kph to meters per second.

$$v_0 = \frac{\left(80\ \frac{\text{km}}{\text{h}}\right)\left(1000\ \frac{\text{m}}{\text{km}}\right)}{3600\ \frac{\text{s}}{\text{h}}} = 22.22\ \text{m/s}$$

In this problem, $v_0 = 22.2$ m/s, $v = 0$, and $s = 95$ m are known. t is the unknown. From Table 55.1,

$$t = \frac{2s}{v_0 + v} = \frac{(2)(95\ \text{m})}{22.22 + 0\ \frac{\text{m}}{\text{s}}} = 8.55\ \text{s}$$

9. LINEAR ACCELERATION

Linear acceleration means that the acceleration increases uniformly with time. Figure 55.4 shows how the velocity and position vary with time.[2]

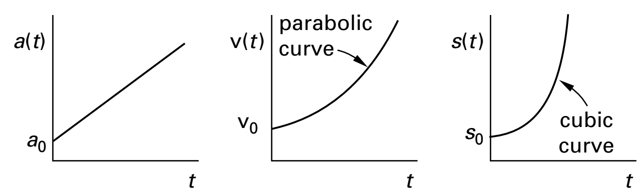

Figure 55.4 *Linear Acceleration*

10. PROJECTILE MOTION

A *projectile* is placed into motion by an initial impulse. (Kinematics deals only with dynamics during the flight. The force acting on the projectile during the launch phase is covered in kinetics.) Neglecting air drag, once the projectile is in motion it is acted upon only by the downward gravitational acceleration (i.e., its own weight). Thus, projectile motion is a special case of motion under constant acceleration.

Consider a general projectile set into motion at an angle of ϕ (from the horizontal plane) and initial velocity v_0. Its range is R, the maximum altitude attained is H, and the total flight time is T. In the absence of air drag, the following rules apply to the case of a level target.[3]

- The trajectory is parabolic.
- The impact velocity is equal to initial velocity, v_0.
- The impact angle is equal to the initial launch angle, ϕ.
- The range is maximum when $\phi = 45°$.
- The time for the projectile to travel from the launch point to the apex is equal to the time to travel from the apex to the impact point.
- The time for the projectile to travel from the apex of its flight path to impact is the same time an initially stationary object would take to fall a distance H.

Table 55.2 contains the solutions to most common projectile problems. These equations are derived from the laws of uniform acceleration and conservation of energy.

Example 55.4

A projectile is launched at 600 ft/sec (180 m/s) with a 30° inclination from the horizontal. The launch point is on a plateau 500 ft (150 m) above the plane of impact. Neglecting friction, find the maximum altitude, H, above the plane of impact, the total flight time, T, and the range, R.

SI Solution

The maximum altitude above the impact plane includes the height of the plateau and the elevation achieved by the projectile.

$$H = z + \frac{v_0^2 \sin^2\phi}{2g} = 150\ \text{m} + \frac{\left(180\ \frac{\text{m}}{\text{s}}\right)^2 (\sin^2 30°)}{(2)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)} = 563\ \text{m}$$

The total flight time includes the time to reach the maximum altitude and the time to fall from the maximum altitude to the impact plane below.

$$\begin{aligned} T &= t_H + t_{\text{fall}} \\ &= \frac{v_0 \sin\phi}{g} + \sqrt{\frac{2H}{g}} \\ &= \frac{\left(180\ \frac{\text{m}}{\text{s}}\right)(\sin 30°)}{9.81\ \frac{\text{m}}{\text{s}^2}} + \sqrt{\frac{(2)(563\ \text{m})}{9.81\ \frac{\text{m}}{\text{s}^2}}} \\ &= 19.9\ \text{s} \end{aligned}$$

[2]Because of the successive integrations, if the acceleration function is a polynomial of degree n, the velocity function will be a polynomial of degree $n+1$. Similarly, the position function will be a polynomial of degree $n+2$.

[3]The case of projectile motion with air friction cannot be handled in kinematics, since a retarding force acts continuously on the projectile. In kinetics, various assumptions (e.g., friction varies linearly with the velocity or with the square of the velocity) can be made to include the effect of air friction.