# Seismic Design of Building Structures

## A Professional's Introduction to Earthquake Forces and Design Details

Ninth Edition

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# 4

# VIBRATION THEORY

## 4-1 TWO APPROACHES TO SEISMIC DESIGN

There are two greatly different approaches to seismic design, both of which are "correct" in their own ways. In a *dynamic analysis*, the overall building and story stiffnesses and rigidities are calculated. (See Sec. 4-4.) A specific design earthquake, including magnitude and loading history, is selected and applied to a mathematical model (consisting of lumped masses, damping, and spring stiffness) of the building. The solution may rest heavily on vibrational theory, finite element analysis, and other advanced structural techniques requiring computer analysis. The response of the system (including the displacement and acceleration functions) is calculated and used to determine the forces in each member as a function of time. This method is almost always used for critical structures such as dams and power plants.

There are a number of factors that can render the dynamic approach inappropriate. The building itself may be too simple or too standardized to warrant the rigorous approach of the design analysis. Conversely, the building may be too complex and have too many degrees of freedom to model mathematically. Also, in the initial design phases, the member sizes and locations may not be known, making it difficult to estimate stiffnesses and rigidities.[1] The dynamic approach is inappropriate, too, when the design earthquake is not known. Additionally, the analysis may be beyond the financial or computational abilities of the engineering firm performing the design. And, finally, unless the building is particularly irregular as defined in ASCE/SEI7 Sec. 12.3.2, there may be no code requirement to perform a dynamic analysis.

The alternative to a dynamic analysis is a *static analysis*. The *equivalent lateral (seismic) force* is calculated as simply some fraction of the dead weight. Chapter 16 of the IBC and Ch. 12 of the ASCE/SEI7 codifies this analysis so that there is no need to know the design earthquake.

In the chapters and sections that follow, these two methods are at times discussed separately and, at other times, aspects of each method are combined. Although considerably different in approach, the static method is based on engineering logic that can, in many cases, be traced back to vibration theory.

## 4-2 SIMPLE HARMONIC MOTION[2]

Ideal vibrational systems that consist of springs and masses and that are not acted upon by external disturbing forces (after an initial displacement) are known as *simple harmonic oscillators.* During steady-state motion, such oscillators move in a repetitive sinusoidal pattern known as *simple harmonic motion.* Simple harmonic motion is characterized by the absence of a continued disturbing force and a lack of frictional damping.

Examples of simple harmonic oscillators are a mass hanging on an ideal spring (Fig. 4.1(a)), a pendulum on a frictionless pivot, and a slab supported on two massless cantilever springs (Fig. 4.1(b)).

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[1] Although there are some real design programs, most "design" programs are actually analysis programs that require the user to input information about the locations and characteristics of the structural members.

[2] There is no need actually to develop the differential equations of oscillatory motion for a building. However, this section introduces some of the terms and concepts related to structural dynamics.

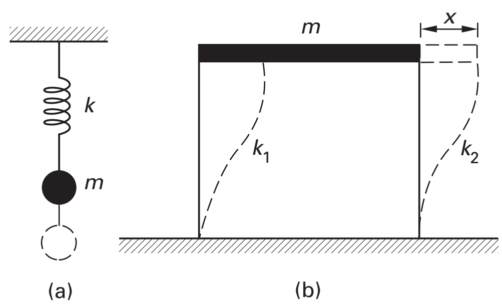



**Figure 4.1** Simple Harmonic Oscillator

The number of variables needed to define the position of all parts of a system is known as its *degrees of freedom.* If the oscillator is constrained to move in one dimension only, or alternatively, if one linear or angular variable is sufficient to describe the position of the oscillator, the system is known as a *single-degree-of-freedom* (SDOF) *system.* The moving mass in an SDOF system is usually concentrated at one point and is known as a *lumped mass.*

Oscillation of the SDOF system shown in Fig. 4.1 is initiated by displacing and releasing the mass. The displacement, $x$, is measured from the equilibrium position. Once the system has been displaced and released, no further external force acts on it. Because there is no friction once it is set in motion, the mass remains in motion indefinitely, as shown in Fig. 4.2.

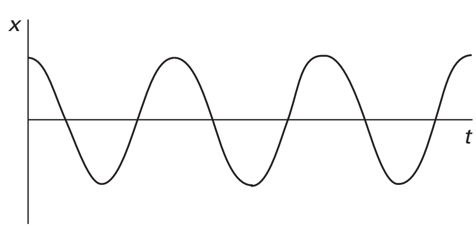



**Figure 4.2** Time Response of an Undamped Simple Harmonic Oscillator

## 4-3 STIFFNESS AND FLEXIBILITY

When a force, $F$, acts on an ideal linear spring, *Hooke's law* predicts the magnitude of the spring deflection, $x$. In Eq. 4.1, $k$ is the *stiffness* or *spring constant* in lbf/ft (N/m). The stiffness is the force that must be applied in order to deflect the spring a distance of one unit.

$$F = kx \quad \text{[Hooke's law]} \qquad [4.1]$$

Referring to the mass-spring system shown in Fig. 4.1(a), the spring is undeflected until the mass is attached to it. After the mass is attached, the spring will deflect an amount known as the *static deflection*, $x_{\text{st}}$.

$$W = \frac{mg}{g_c} = kx_{\text{st}} \qquad \text{[U.S.]} \quad [4.2(a)]$$

$$W = mg = kx_{\text{st}} \qquad \text{[SI]} \quad [4.2(b)]$$

The stiffness, $k$, of a beam can be calculated as the ratio of applied force to deflection from the beam deflection tables that are typically in every mechanics of materials textbook. Table 4.1 summarizes some of these terms.

*Compliance (flexibility)* is the reciprocal of stiffness. It is the deflection obtained when a unit force is applied. Therefore, its units are ft/lbf (m/N).

## 4-4 RIGIDITY

Strictly speaking, *rigidity*, $R$, is the reciprocal of deflection. In buildings where all members consist of the same material and all walls have the same thickness (for example, a masonry-walled building or an all-concrete building), the deflection is traditionally calculated with arbitrary values of applied force, modulus of elasticity, and wall thickness. This is permitted when distributing the applied lateral loads to vertical members because the load "taken" by each member is proportional to the member's *relative rigidity.*

$$R = \frac{1}{x_{\text{in}}} \qquad \text{[U.S.]} \quad [4.3(a)]$$

$$R = \frac{25.4}{x_{\text{mm}}} \qquad \text{[SI]} \quad [4.3(b)]$$

Both moment and shear contribute to the deflection experienced by a vertical member (e.g., a shear wall).[3] Consider the wall shown in Fig. 4.3(a). This wall is fixed at the top and bottom and bends in double curvature since the top and bottom must remain vertical. Such a wall is known as a *fixed pier.* The deflection due to both shear and flexure of a fixed pier is given by Eq. 4.4.

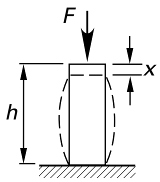
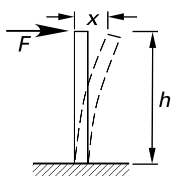
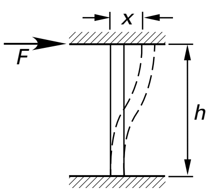
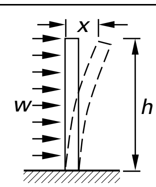
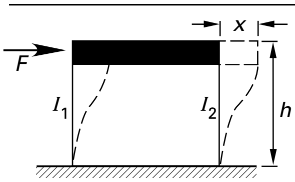
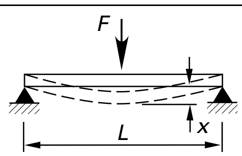
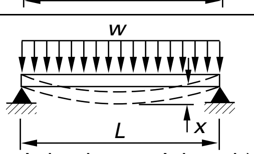
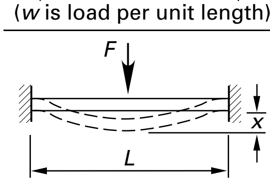
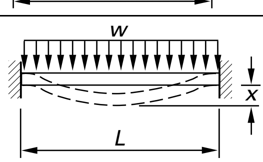
$$x_{\text{fixed}} = \frac{Fh^3}{12EI} + \frac{1.2Fh}{AG} \quad \text{[fixed pier]} \qquad [4.4]$$

$$A = td \qquad [4.5]$$

$$I = \frac{td^3}{12} \qquad [4.6]$$

[3]Common beam deflection equations, such as those presented in Table 4.1, usually disregard the effect of shear. However, shear contributes to deflection when the ratio of height to depth is low. In general, shear deflection should not be neglected, unless beam spans are long.

**Table 4.1**
Deflection and Stiffness for Various Systems
(Due to Bending Moment Alone)

| system | maximum deflection ($x$) | stiffness ($k$) |
|---|---|---|
|  | $\frac{Fh}{AE}$ | $\frac{AE}{h}$ |
|  | $\frac{Fh^3}{3EI}$ | $\frac{3EI}{h^3}$ |
|  | $\frac{Fh^3}{12EI}$ | $\frac{12EI}{h^3}$ |
|  | $\frac{wh^4}{8EI}$ | $\frac{8EI}{h^3}$ |
|  | $\frac{Fh^3}{12E(I_1+I_2)}$ | $\frac{12E(I_1+I_2)}{h^3}$ |
|  | $\frac{FL^3}{48EI}$ | $\frac{48EI}{L^3}$ |
|  (*w* is load per unit length) | $\frac{5wL^4}{384EI}$ | $\frac{384EI}{5L^3}$ |
|  | $\frac{FL^3}{192EI}$ | $\frac{192EI}{L^3}$ |
|  (*w* is load per unit length) | $\frac{wL^4}{384EI}$ | $\frac{384EI}{L^3}$ |

A wall that is fixed at the bottom, but free to rotate at the top, bends in simple curvature and is known as a *cantilever pier*. The deflection of a cantilever wall due to both effects is

$$x_{\text{cantilever}} = \frac{Fh^3}{3EI} + \frac{1.2Fh}{AG} \quad \text{[cantilever pier]} \tag{4.7}$$

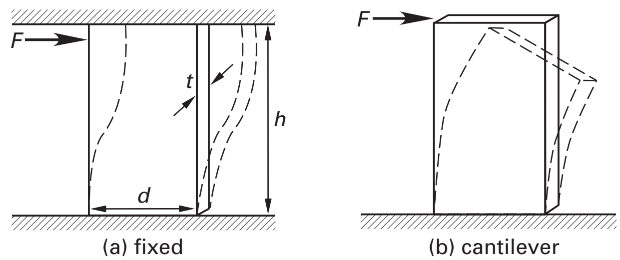



**Figure 4.3** Fixed and Cantilever Piers

For concrete, $E \approx 3 \times 10^6$ psi ($2.1 \times 10^7$ kPa) and $G \approx 0.4E$. For masonry, $E \approx 1 \times 10^6$ psi ($6.9 \times 10^6$ kPa) and $G \approx 0.4E$. For steel, $E \approx 3 \times 10^7$ psi ($2.1 \times 10^8$ kPa) and $G \approx 1.2 \times 10^7$ psi ($8.3 \times 10^7$ kPa). For wood, $E \approx 1 \times 10^6$ psi to $1.8 \times 10^6$ psi ($6.9 \times 10^6$ kPa to $12 \times 10^6$ kPa). However, since the shear that is distributed to each vertical member (i.e., each pier) is in proportion to the relative rigidity and does not depend on the actual rigidity, the deflections can be calculated with arbitrary values of total shear, $F$, and wall thickness, $t$. Equations 4.8 and 4.9 use $F$ = 100,000 lbf (445 000 N), $t$ = 1.0 in (25 mm), $E = 1 \times 10^6$ psi ($6.9 \times 10^6$ kPa), and arbitrary units.

$$R_{\text{fixed}} = \frac{1}{0.1\left(\frac{h}{d}\right)^3 + 0.3\left(\frac{h}{d}\right)} \tag{4.8}$$

$$R_{\text{cantilever}} = \frac{1}{0.4\left(\frac{h}{d}\right)^3 + 0.3\left(\frac{h}{d}\right)} \tag{4.9}$$

## 4-5 NATURAL PERIOD AND FREQUENCY

The time for a complete cycle of oscillation of an SDOF system is known as the *fundamental* or *natural period*, $T$, usually expressed in seconds. The reciprocal of natural period is the *linear natural frequency*, $f$, usually called *natural frequency*, *fundamental frequency*, or just *frequency*, and expressed in Hz (i.e., cycles per second). It is important to distinguish between the natural frequency of a system (building, oscillator, etc.) and the frequency of an applied force. The natural frequency, $f$, in Eq. 4.10 has nothing to do with an external force.

$$f = \frac{1}{T} \tag{4.10}$$

The natural frequency can also be expressed in radians per second (rad/sec), in which case it is known as the *circular frequency*, *angular natural (fundamental) frequency*, or just *angular frequency*, $\omega$.

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad [4.11]$$

It is easy to derive the natural frequency for the case of a simple harmonic oscillator.[4] For a mass on a spring,

$$\omega = \sqrt{\frac{kg_c}{m}} = \sqrt{\frac{kg}{W}} \qquad \text{[U.S.]} \quad [4.12(a)]$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{[SI]} \quad [4.12(b)]$$

Substituting $k$ from Hooke's law (Eq. 4.1) and recognizing that the mass, $m$, can be calculated from the weight, $W$, an expression is derived for the natural frequency in terms of the static deflection, $x_{\text{st}}$, calculated in Sec. 4-3.

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Fg_c}{x_{\text{st}}m}} = \sqrt{\frac{Fg}{x_{\text{st}}W}} \qquad \text{[U.S.]} \quad [4.13(a)]$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{F}{x_{\text{st}}m}} \qquad \text{[SI]} \quad [4.13(b)]$$

Since Eq. 4.13 can be used to calculate the natural period, it is tempting to substitute the maximum allowable code drift (i.e., 2.5% of the total building height; see Sec. 6-40) for the static deflection in order to calculate the natural building period.[5] Such a substitution would require no structural analysis at all, but implies that the building will have maximum flexibility permitted by the code and will remain elastic when this drift is achieved. One problem with this approach is that it assumes the maximum allowable drift to be the same for all geographic regions, although the flexibility actually depends on the location since flexibility is affected by the building's seismic resistance. Thus, while the lateral forces on the building differ, the maximum drift and, thus, the period, do not. Obviously, the building period cannot be calculated in this way.

[4]This is done in virtually every physics, dynamics, and earthquake book, but not here.

[5]The IBC permits the drift limits to be exceeded when it is demonstrated that greater drift can be tolerated by both structural elements and nonstructural elements.

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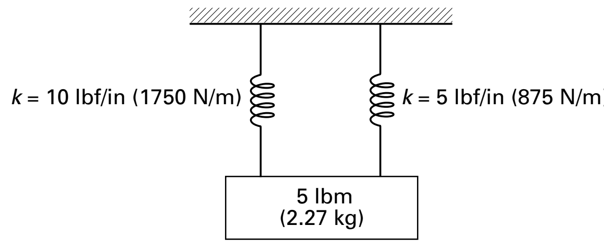
*Example 4.1*

A 5 lbm (2.27 kg) mass hangs from two ideal springs as shown. The block is constrained so that it does not rotate. What is the natural period of vibration?




*Customary U.S. Solution*

Both springs must deflect in order for the mass to move. The total composite spring constant is

$$\begin{aligned} k_t &= k_1 + k_2 = \left(5\,\frac{\text{lbf}}{\text{in}} + 10\,\frac{\text{lbf}}{\text{in}}\right)\left(12\,\frac{\text{in}}{\text{ft}}\right) \\ &= 180\,\text{lbf/ft} \end{aligned}$$

From Eqs. 4.11 and 4.12, the natural period is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{g_c k}} \\ &= 2\pi\sqrt{\frac{5\text{ lbm}}{\left(32.2\,\frac{\text{ft-lbm}}{\text{lbf-sec}^2}\right)\left(180\,\frac{\text{lbf}}{\text{ft}}\right)}} \\ &= 0.185\text{ sec} \end{aligned}$$

*SI Solution*

Both springs must deflect in order for the mass to move. The total composite spring constant is

$$\begin{aligned} k_t &= k_1 + k_2 = 1750\,\frac{\text{N}}{\text{m}} + 875\,\frac{\text{N}}{\text{m}} \\ &= 2625\,\text{N/m} \end{aligned}$$

From Eqs. 4.11 and 4.12, the natural period for vertical translation is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.27\text{ kg}}{2625\,\frac{\text{N}}{\text{m}}}} \\ &= 0.185\text{ s} \end{aligned}$$

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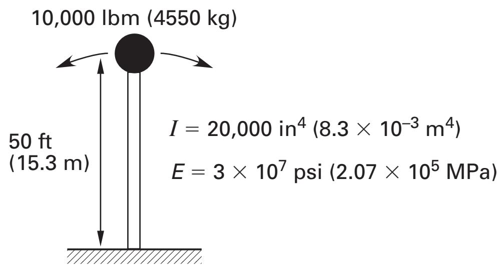
*Example 4.2*

A small water tank is supported on a slender column as shown. Neglecting the weight of the column, calculate the natural period of vibration.




*Customary U.S. Solution*

Consider the water tower to be a cantilever beam. The stiffness is the force required to deflect the tank 1 ft laterally.

From Table 4.1,

$$k = \frac{3EI}{h^3} = \frac{(3)\left(3\times 10^7\,\frac{\text{lbf}}{\text{in}^2}\right)(20{,}000\text{ in}^4)}{(50\text{ ft})^3\left(12\,\frac{\text{in}}{\text{ft}}\right)^2}$$
$$= 1\times 10^5\text{ lbf/ft}$$

From Eqs. 4.11 and 4.12, the period is

$$T = 2\pi\sqrt{\frac{m}{g_c k}}$$
$$= 2\pi\sqrt{\frac{10{,}000\text{ lbm}}{\left(32.2\,\frac{\text{ft-lbm}}{\text{lbf-sec}^2}\right)\left(1\times 10^5\,\frac{\text{lbf}}{\text{ft}}\right)}}$$
$$= 0.35\text{ sec}$$

*SI Solution*

Consider the water tower to be a cantilever beam. The stiffness is the force required to deflect the tank 1 m laterally.

From Table 4.1,

$$k = \frac{3EI}{h^3}$$
$$= \frac{(3)(2.07\times 10^5\text{ MPa})\left(10^6\,\frac{\text{Pa}}{\text{MPa}}\right)(8.3\times 10^{-3}\text{ m}^4)}{(15.3\text{ m})^3}$$
$$= 1.44\times 10^6\text{ N/m}$$

From Eqs. 4.11 and 4.12, the natural period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{4550\text{ kg}}{1.44\times 10^6\,\frac{\text{N}}{\text{m}}}}$$
$$= 0.35\text{ s}$$

## 4-6 DAMPING

*Damping* is the dissipation of energy from an oscillating system, primarily through friction. The kinetic energy is transformed into heat. All structures have their own unique ways of dissipating kinetic energy, and in certain designs, mechanical systems known as *dampers* (see Sec. 14-4) can be installed to increase the damping rate.[6]

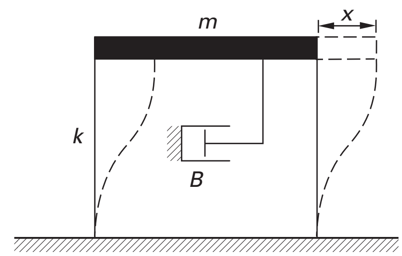



**Figure 4.4** Oscillator with Damping

There are several sources of damping. *External viscous damping* is caused by the structure moving through surrounding air (or water, in some cases). It is generally small in comparison to other sources of damping. *Internal viscous damping*, commonly the only type of damping actually modeled, is related to the viscosity of the structural material. It is proportional to velocity. (See Eq. 4.14.) *Body-friction damping*, also known as *Coulomb friction*, results from friction between members in contact. It includes friction at connection points. Sections of opposed cracked masonry walls rubbing back and forth against one another are very effective body-friction dampers. Another source of damping, *radiation damping*, occurs as a structure vibrates and becomes a source of energy itself. Some of the energy is reradiated

[6]A damper is similar in design to a shock absorber and is often depicted as a plunger moving through a pot of viscous fluid. In modeling, dampers are also known as *dashpots*, although this term is more common among mechanical engineers.